**Calculus III for CS Project Write-ups**

1. The Hilbert Matrix

1) The charts below summarize the errors for LU, Householder and Givens factorizations. The errors have been plotted as a function of *n*. All the errors are very small. Givens appears to have an outlier data point at n = 14. Aside from that, the general trend is that Householder reflections are the least accurate, followed by Givens rotations, followed by LU factorizations.

1. **Why is it justified to use the LU or QR-factorizations as opposed of calculating an inverse matrix?**

By using LU or QR-factorizations, we can solve the problem with forward and backward substitution. This process involves several numbers of simple algebraic operations directly proportional to the size of the given matrix. On the other hand, calculating the inverse matrix would not be easily formulated and would not be as accurate. If one did create such a procedure, it would require many more multiplication and division operations, which would increase the final error due to the finite precision of Double floating points.

ii) **What is the benefit of using LU or QR-factorizations in this way?**

Using LU or QR-factorizations does not incur significant error, as seen in the graphs above. As n increases, there is little change in the error of (LU – H), (QR – H), and (Hx – b), meaning that these factorizations can be used effectively when scaled to higher n values. The increase in error, which was recorded to be no larger than 1E-12 for (Hx - b), is worth it due to a decrease in runtime. The conditioning error is minimized as much as possible. Conditioning is a property of the matrix A that determines whether it is feasible for an algorithm to provide a numerical solution to a linear system involving A. Since any linear system on a computer has some error, it is important to reduce perturbation or imprecision as much as possible. LU and QR help to accomplish this while giving increased speeds in calculations.

1. Convolution Codes
   1. **For the written component of this part, compare the results of the two methods above, and discuss the number of iterations required to obtain the desired precision.**

When inputting matrix

and tolerance = 10-3, the Jacobi iteration method took 8 iterations while the Gauss-Seidel took only 6 to reach the desired precision. This result suggests that the Gauss-Seidel is the faster iteration method.

* 1. **Is the length of the initial stream n important? With respect to number of iterations?**

The length of the initial stream n is not important, even with respect to the number of iterations.

* 1. **Does n have an effect on the number of iterations required to achieve the error tolerance?**

No. It does not appear to be important. The number of iterations does not increase for n = 5, 10, 15, 20, or 25. The iterations remain consistently the same.

**Urban Population Dynamics**

**1) - Interpret the data in the matrix, and discuss the social factors that influence those numbers.**

**Examining the Properties of a Leslie Matrix**

The Leslie Matrix describes the growth of populations and projected age distributions of an initial population into following generation. The matrix has the following properties:

1. Each column of the matrix contains data about a certain age group of the population.
2. The first row of each column of data in the Leslie Matrix contains data about the fecundity*, fx*, or the average number of female offspring reaching the next generation born from a mother of the current generation from that age group.
3. Below the first row, all elements in that column are zero except for one unique row. None of the other columns in the matrix will have data in that row since those columns will represent a different group, and that row is linked to a particular group. The data in this unique row represents a survival fraction for that portion of the population to go onto the next generation. This fraction is the percentage of individuals to survive one generation to the next.

For example, take the first vector (column zero) of the given Leslie Matrix. It contains the data [0, 0.7, 0, 0, 0, 0, 0, 0, 0] t.

1. The first element contains *fx*, the fertility of this portion of the population.
2. The second element is nonzero, namely 0.7, so it is the unique element whose data represents a survival fraction of the population.
   1. Given a vector v whose data represents the population at a certain date, only the first element will be multiplied by this survival fraction (0.7) and carried into the next generation. This means that 70% of the existing population of the first vector’s age class will be carried forth into the next generation. This process continues and is the same for all age classes.

**Analyzing the Data in the Leslie Matrix**

As the given Leslie Matrix is used to model human population growth in a city, with each column separating the population by age group, there are many factors that affect the data in its elements.

1. We can consider different age classes to correlate with different fecundities and survival fractions.
   1. Higher aged groups contain elder individuals who are less likely to be alive 10 years afterwards as they approach their life expectancies. This makes them less likely to survive into the next decade/generation of the population, and is mathematically represented with smaller fractions inside the Leslie Matrix (e.g. 0.4).
   2. In contrast, smaller and middle aged groups contain individuals who are more fit, physically able, and mentally healthy than their elder counterparts. Thus, they have larger survivability fractions (e.g. 0.85, 0.9, 0.88).
   3. However, fewer people survive from one generation to the next in their youngest years (0-10) than in their middle aged years (20-40) because of factors that may affect one age group that do not affect one another.
      1. For example, children who obtain medical complications during birth are likely to manifest themselves early in a child’s life and potentially cut a younger individual’s life short. A middle aged individual is less likely to die of such medical complications from birth because he or she has already survived for so long without any serious side effects of such complications.
      2. Children, adolescents, and college students as a group experience significantly more social and academic stress than their middle aged counterparts. This is because students are often struggling to understand changes in their bodies like imbalances of hormones, while dealing with work of increasing difficulty and attempting to maintain relationships with others. In addition, these younger age groups eat less nutritiously as a whole as their easiest access to food is often the least healthy, as compared to an adult who has the freedom and ability to cook. These stresses contribute to a lower survivability fractions from one generation to the next.
2. We can consider the fecundity of each age group as dependent on the age class as well.
   1. Conceivably, there are no 0-10 year olds who are capable of producing children so *fx* is 0 for them.
   2. On the other side of the spectrum of age classes, fertility decreases with age past the middle years, so *fx* declines from 0.9 for 30-40 age class to 0.1 for the 40-50 age class to 0 for the 50-60 age class. The older age classes have little to no chance of producing offspring at their ages to pass on to the next generation as their bodies are past child-rearing age.
   3. Middle aged groups have a much greater chance of producing 1 or even more than 1 children, as represented by numbers greater than 1 in the first row of the matrix. Humans are biologically most fertile at these middle age, so it makes sense that most people would be having children at this age. In addition, these are the years that adolescents become adults and lead more financially stable lives, often marrying to form family units and obtaining jobs. Thus they are most able to provide for a child and nourish it to survive to the next generation.

**2 - What will the population distribution be in 2010? 2020? 2030? 2040? 2050? Calculate also the total population in those years, and by what fraction the total population changed each year.**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Data/Year | 2000 | 2010 | 2020 | 2030 | 2040 | 2050 |
| Population Distribution Vector | [  210000, 190000, 180000, 210000, 200000, 170000, 120000, 90000, 50000  ]^t | [  635000, 147000, 161500, 162000, 189000, 176000, 136000, 92400, 36000  ]^t | [  518750, 444500, 124950, 145350, 145800, 166320, 140800, 104720, 36960  ]^t | [  816240, 363125, 377825, 112455, 130815, 128304, 133056, 108416, 41888  ]^t | [ 965648.5, 571368, 308656.25, 340042.5, 101209.5, 115117.2, 102643.2, 102453.12, 43366.4  ]^t | [ 1341322.675, 675953.95, 485662.8, 277790.625, 306038.25, 89064.36, 92093.76, 79035.264, 40981.248  ]^t |
| Total Population | 1210000 | 1099900 | 1309400 | 1395884 | 1684856 | 2046620 |
| Fraction Change Total Population from Previous (%) | N/A | -9.099 | 19.047 | 6.605 | 20.702 | 21.472 |

The population distribution vector specifies the population of each age class by element. The first element contains age class 0-10, the second element contains age class 10-20, and so on.

The total population was calculated by simply adding all individual populations for each age class in a given year.

The fractional change was calculated as (current population – previous population) / previous population \* 100

**3 - Use the power method to calculate the largest eigenvalue of the Leslie matrix A. The iteration of the power method should stop when you get 8 digits of accuracy. What does this tell you? Will the population go to zero, become stable, or be unstable in the long run? Discuss carefully and provide the mathematical arguments for your conclusion. You might want to investigate the convergence of ||Ak||.**

Eigenvalue: .99999999, or approximately 1

This means that the population will become stable in the long run, after a period of growth. It appears to converge exactly to one, telling us that as time approaches infinity, the population should approximately stay the same. The population will not go to zero. An eigenvalue is defined such that Ax = λx. If λ is less than 1, this means the population vector is being scaled down and decreasing in magnitude as a result of the matrix A being multiplied by it. If λ is greater than 1, then the population vector is being scaled up and is increasing in magnitude as a result of the matrix A being multiplied by it. If λ is 1, then the population vector stays constant.

**4 - Suppose we are able to decrease the birth rate of the second age group by half in 2020. What are the predictions for 2030, 2040 and 2050? Calculate again the largest eigenvalue of A (to 8 digits of accuracy) with your program and discuss its meaning regarding the population in the long run.**

2030: [549540.0, 363125.0, 377825.0, 112455.0, 130815.0, 128304.0, 133056.0, 108416.0, 41888.0]^t

2040: [747773.5, 384678.0, 308656.25, 340042.5, 101209.5, 115117.2, 102643.200000, 102453.12, 43366.4]^t

2050: [886487.875, 523441.4499910, 326976.3, 277790.625, 306038.25, 89064.36, 92093.7600000, 79035.2640000, 40981.248]^t

Eigenvalue:  1.167902719

A decrease in the birth rate of the second age by a half changed the calculated eigenvalue to 1.167902719, which is still very close to one but is greater. This means that the population went to stable growth since the eigenvalue is greater than one. This means that the population increases as t approaches infinity.